

NONNEGATIVE RANK DEPENDS ON THE FIELD

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ABSTRACT. Let A be a nonnegative matrix with entries from a field $\mathcal{F} \subset \mathbb{R}$. The nonnegative rank of A , denoted $\text{rank}_+(A, \mathcal{F})$, is the smallest integer k such that A is a sum of k nonnegative rank-one matrices over \mathcal{F} . We show that, for some A and \mathcal{F} , it holds that $\text{rank}_+(A, \mathcal{F}) \neq \text{rank}_+(A, \mathbb{R})$.

In 1993, Cohen and Rothblum published the foundational paper [2], which contained a description of possible applications of nonnegative rank and an algorithm computing this function. Their algorithm was based on quantifier elimination procedure, and the authors did not know whether it is applicable to compute nonnegative ranks over fields which are not real closed. Cohen and Rothblum posed the question: How sensitive is the problem to the ordered field over which it is asked?

Problem 1. Do there exist a subfield $\mathcal{F} \subset \mathbb{R}$ and a nonnegative matrix $A \in \mathcal{F}^{m \times n}$ such that $\text{rank}_+(A, \mathcal{F}) \neq \text{rank}_+(A, \mathbb{R})$?

Now the nonnegative matrix factorization is one of the important problems in applied mathematics. It is a key tool of new techniques developed in contexts of data processing [6], combinatorial optimization [3], statistics [5], and several other topics. The computational complexity of nonnegative rank is much better understood now [7], and faster algorithms for solving this problem have been developed [1]. However, Problem 1 remained open. The paper [4] gives a positive solution of a similar problem asked for *positive semidefinite* rank. Kubjas, Robeva, and Sturmfels [5] stressed the importance of describing matrices of nonnegative rank at most r with quantifier-free semialgebraic formulae. In the special case of rank-three matrices, they solved Problem 1 in the negative and constructed such formulae.

In this paper, we give a positive solution of Problem 1. In particular, we will see that quantifier-free semialgebraic formulae cannot describe matrices of nonnegative rank at most five over a specific field. Many known algorithms computing nonnegative ranks are based on quantifier elimination procedures [1, 2], and our result shows that they cannot be used for matrices over arbitrary ordered field. We start with a geometric reformulation of Problem 1; its connection with the following question is established by standard arguments, but we list the proof for completeness.

Problem 2. Do there exist a field $\mathcal{F} \subset \mathbb{R}$ and polytopes $P, Q \subset \mathbb{R}^{k-1}$ such that:

- (1) $\dim P = k - 1$ and $P \subset Q$;
- (2) vertices of both P and Q have coordinates from \mathcal{F} ;
- (3) there is a simplex Δ satisfying $P \subset \Delta \subset Q$, and every such Δ has a vertex not all of whose coordinates belong to \mathcal{F} ?

Proposition 3. *A positive solution of Problem 2 implies that of Problem 1.*

Proof. Denote by d_1, \dots, d_k the vertices of Δ , by v_1, \dots, v_m the vertices of P , and let $f_1 \geq 0, \dots, f_n \geq 0$ be linear inequalities defining the facets of Q . Let A be an

n -by- m matrix whose i th column is the *slack* of v_i , that is, define A_{ij} as $f_j(v_i)$. Since every v_i belongs to the convex hull of d_1, \dots, d_k , the i th column of A belongs to the convex hull of the slacks of d_1, \dots, d_k .

Conversely, assume that $A = BC$, for some nonnegative matrices $B \in \mathcal{F}^{n \times k}$ and $C \in \mathcal{F}^{k \times m}$. Then, since $\dim P = k - 1$ implies $\text{rank}(A) = k$, we see that $\text{rank}(B) = k$. Therefore, the columns of B (up to scaling) are affine combinations of columns of A . Therefore, every column of B is the slack vector of some point of \mathbb{R}^{k-1} , and this point belongs to Q because B is nonnegative. \square

Now we describe our solution of Problem 2; all the computational claims have been checked using *Wolfram Mathematica*. We define $\alpha = -0.0311\dots$, $\beta = -0.4088\dots$, $\gamma = 0.3983\dots$ as numbers that make the polynomial $\pi(t) = (-96\alpha - 96\beta - 48\alpha\beta - 192\gamma - 96\alpha\gamma - 96\beta\gamma + 99\alpha\beta\gamma) + t(-1392\alpha - 1296\beta - 96\alpha\beta - 1696\gamma - 1624\alpha\gamma - 952\beta\gamma + 1370\alpha\beta\gamma) + t^2(-256 - 8608\alpha - 5488\beta + 1172\alpha\beta - 5536\gamma - 10240\alpha\gamma - 2164\beta\gamma + 4292\alpha\beta\gamma) + t^3(-2944 - 25648\alpha - 8512\beta + 4100\alpha\beta - 8192\gamma - 21056\alpha\gamma - 1736\beta\gamma + 4768\alpha\beta\gamma) + t^4(-8576 - 32288\alpha - 4832\beta + 4072\alpha\beta - 2912\gamma - 12608\alpha\gamma - 440\beta\gamma + 1744\alpha\beta\gamma) + t^5(-3584 - 11648\alpha - 896\beta + 1120\alpha\beta) \in \mathbb{Q}[t]$ have the form $(-t-2)u^2$, where $u \in \mathbb{R}[t]$ has degree two; we pick $\tau = 0.1765\dots$ to be one of the two roots of u . We denote $\mathcal{F} = \mathbb{Q}(\alpha, \beta, \gamma)$; it can be checked that $\tau \notin \mathcal{F}$.

Our construction is four-dimensional, and we work in the affine subspace $x_1 + \dots + x_5 = 1$ in \mathbb{R}^5 ; in particular, we refer to three-dimensional planes in this subspace as hyperplanes. We define $\Omega(t) = \mu(1+t, 1+2t, 1+t, 1, 0)$, $F_{11} = \lambda_{11}(1000 + 514\alpha, 1056, 524 + 131\alpha, 772 + 193\alpha, 648 + 162\alpha)$, $F_{12} = \lambda_{12}(1000 + 532\alpha, 1128, 1012 + 253\alpha, 236 + 59\alpha, 624 + 156\alpha)$, $F_{13} = \lambda_{13}(500 + 233\alpha, 432, 536 + 134\alpha, 176 + 44\alpha, 356 + 89\alpha)$, $F_{21} = \lambda_{21}(14044 + 3511\beta, 20000 + 9467\beta, 17868, 8888 + 2222\beta, 19200 + 4800\beta)$, $F_{22} = \lambda_{22}(136 + 34\beta, 200 + 98\beta, 192, 144 + 36\beta, 128 + 32\beta)$, $F_{23} = \lambda_{23}(96 + 24\beta, 100 + 28\beta, 12, 72 + 18\beta, 120 + 30\beta)$, $F_{31} = \lambda_{31}(26 - 3\gamma, 22 - 2\gamma, 25 + 5\gamma, 20, 7)$, $F_{32} = \lambda_{32}(17 - 3\gamma, 22 - 2\gamma, 25 + 5\gamma, 20, 16)$, $F_{33} = \lambda_{33}(376 - 81\gamma, 384 - 54\gamma, 500 + 135\gamma, 540, 200)$, $F_{41} = \lambda_{41}(618, 392, 365, 625, 500)$, $F_{42} = \lambda_{42}(1863, 1252, 1250, 1875, 1260)$, $F_{43} = \lambda_{43}(384, 496, 495, 625, 500)$, $H = (3, 3, 3, 3, 4)/16$, where the coefficients are such that the coordinates of corresponding vectors sum to 1.

We write ε for a sufficiently small positive number in the rest of this note. We choose rational numbers q_1, q_2 satisfying $\tau \in (q_1, q_2)$ and $|q_1 - q_2| < \varepsilon$, and we define $\Omega = \Omega(\tau)$ and $\Omega_i = \Omega(q_i)$. We let ω be a point satisfying $\|\Omega - \omega\| < \varepsilon$, and we set $f_{ij} = (F_{ij} - (0, 0, 0, 0, v_{ij}))/ (1 - v_{ij})$, where v_{ij} are variables. By $v = (v_{ij})$ we denote the list of all v_{ij} 's. By $\pi_i(\omega, v)$ we denote a nonzero vector orthogonal to $\omega, f_{i1}, f_{i2}, f_{i3}$; then, $\pi_i(\omega, v) \cdot x = 0$ is the equation of the hyperplane passing through those points. We define $V_j(\omega, v)$ as the point that lies on the intersection of hyperplane $x_j = 0$ and all hyperplanes $\pi_i(\omega, v) \cdot x = 0$ with i different from j . We write $V_j = V_j(\Omega, 0)$.

We define $\Delta(\omega, v)$ as the convex hull of ω and all $V_i(\omega, v)$'s, and also $\Delta = \Delta(\Omega, 0)$. We refer to the facet of $\Delta(\omega, v)$ not containing $V_i(\omega, v)$ as the i th facet. Further, we choose rational interior points $W, W_1, \dots, W_4 \in \Delta$ satisfying $\|W - \Omega\| < \varepsilon$ and $\|W_i - V_i\| < \varepsilon$. We define P as the convex hull of W, W_i 's and F_{ij} 's. We also define A_i, B_i, C_i, D_i as arbitrary points with zero i th coordinate such that $\text{conv}\{A_i, B_i, C_i, D_i\}$ has diameter less than ε and contains V_i as an interior point. We define Q as the convex hull of Ω_1, Ω_2 and all A_i 's, B_i 's, C_i 's, D_i 's, and we check that $P \subset \Delta \subset Q$. Also, we check that f_{ij} lies outside Δ if and only if $v_{ij} > 0$. We consider the function $\Psi(\omega, v) = \det(V_1(\omega, v), V_2(\omega, v), V_3(\omega, v), V_4(\omega, v), H)$, and we

check that $\Psi(\Omega, 0) = 0$ and $\det(V_1, V_2, V_3, V_4, \Omega) > 0$. The latter condition implies that H lies outside $\Delta(\omega, v)$ whenever ω and small v_{ij} 's are such that $\Psi(\omega, v) < 0$.

Now let Δ' be a simplex; we are going to complete our proof by checking that $P \subset \Delta' \subset Q$ implies $\Delta = \Delta'$. The positions of W_i 's guarantee that Δ and Δ' get close one to the other as ε goes to zero, so we can assume that ω is a vertex of Δ' . Since Δ and Δ' are close, there is no j such that the i th facet of Δ' is parallel to the line containing all f_{ij} 's; therefore, there are small v_{ij} for which the i th facet of Δ' passes through points f_{ij} . Since $P \subset \Delta'$, the point F_{ij} cannot lie outside Δ' ; therefore, the point $f_{ij} \in \Delta'$ cannot be interior for Δ , and we have $v_{ij} \geq 0$. Now we see that v_{ij} 's and ω are sufficient to determine the hyperplanes of all facets containing ω . Without a loss of generality we can assume that the vertices of Δ' belong to the boundary of Q ; that is, we can assume that the vertices of Δ' are $v_i(\omega, v)$'s. In other words, we have $\Delta' = \Delta(\omega, v)$.

It remains to prove that $(\Omega, 0)$ is a local maximum of the function $\Psi(\omega, v)$ subject to $\omega \in Q$ and $v_{ij} \geq 0$. We can check that partial derivatives $\partial\Psi/\partial v_{ij}$ are negative in the point $(\Omega, 0)$, as well as the derivatives along the directions from $(\Omega, 0)$ to $(A_i, 0)$, $(B_i, 0)$, $(C_i, 0)$, $(D_i, 0)$. Now it suffices to prove that Ω is a local maximum of the function $\Psi(\omega, 0)$ subject to $\omega \in \text{conv}\{\Omega_1, \Omega_2\}$. In this case, Ψ is a rational function $\varphi(t)$; this function factors as $\varphi(t) = \zeta(t)\pi(t)$, where ζ takes a non-zero value in τ , and π is a polynomial defined above. Therefore, the chosen values of $\alpha, \beta, \gamma, \tau$ are such that $\varphi(t)$ has a double root in τ , and it remains to check that $\varphi''(\tau) < 0$.

One particular implication of our result is that algorithms based on quantifier elimination may not help to compute nonnegative factorizations over different fields. One can ask: Is there an algorithm computing $\text{rank}_+(A, \mathbb{Q})$, given a rational non-negative matrix A ? A more specific question asked by Cohen and Rothblum [2] is also open: Is the equality $\text{rank}_+(A, \mathbb{R}) = \text{rank}_+(A, \mathbb{Q})$ always true?

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